Gauge theory of flows of an ideal fluid

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Abstract: Fluid mechanics is a field theory of mass flows of Galilean symmetry. In the gauge theory of theoretical physics, a guiding principle is that laws of physics should be expressed in a form that is independent of any particular coordinate system. Variational formulations of fluid mechanics are reviewed from this point of view, and then a new variational formulation is proposed, which leads to a new representation of compressible rotational flows of an ideal fluid. This improves the classical solution of Clebsch (1859). Present Lagrangian for the action principle consists of main terms of total kinetic energy and internal energy (with negative sign), together with three additional terms yielding the equations of continuity, entropy and the third term which provides rotational component of velocity field. It is verified that the system of new expressions in fact satisfies the Euler's equation of motion. Associated with two symmetries (translation and space-rotation), there are two gauge fields: $E \equiv (v \cdot \nabla)v$ and $H \equiv \nabla \times v$, which do not exist in the system of discrete masses. One can show that those are analogous to the electric field and magnetic field in the electromagnetism, and fluid Maxwell equations can be formulated for E and H. Sound wave within the fluid is analogous to the electromagnetic wave, in the sense that phase speeds of both waves are independent of wave lengths, i.e. non-dispersive.

Key-Words: Gauge theory, Ideal fluid, Variational formulation, New solution, Fluid Maxwell equations

1 Introduction

A symmetry of a physical system means invariance with respect to a certain group of transformations and plays an essential role in the gauge theory of theoretical physics. Fluid mechanics is a field theory of Newtonian mechanics of Galilean symmetry. It is wellknown that a system of *point masses* has global gauge symmetries with respect to both translation and rotation. In traditional mechanics, these are interpreted as the homogeneity and isotropy of the space (e.g. in Landau & Lifshitz (1976) [1], $\S7$ and $\S9$), which are in fact the symmetries yielding conservation laws of total momentum and angular momentum respectively. Corresponding to those symmetries of the mechanical system of discrete masses, two symmetries of fluid flows are known: symmetries of translation (space and time) and space-rotation.

Guided by the gauge theory, Kambe [2, 3] studied flow fields of an ideal compressible fluid and investigated consequence of both global and local invariances of the fields in the space-time (x, t), where $x = (x^1, x^2, x^3)$ is the three-dimensional space coordinates. Velocity field is represented as v(x, t). An essential building block of the gauge theory is the *covariant derivative*. In fact, the convective derivative D_t , defined by

 $\mathbf{D}_t \equiv \partial_t + \boldsymbol{v} \cdot \nabla, \quad \partial_t \equiv \partial/\partial t, \quad \nabla = (\partial/\partial x^i),$ (1)

(*i.e.* the Lagrange derivative in the fluid mechanics) can be identified as the covariant derivative in the framework of gauge theory. Based on this, we can define appropriate Lagrangian functions for motions of an ideal fluid. Euler's equation of motion can be derived from the variational principle, *i.e.* the action principle. In addition, the continuity equation and entropy equation are derived simultaneously.

Standard variational formulations of fluid flows are reviewed first, and then a new variational formulation is proposed, which leads to a new representation of compressible rotational flows of an ideal fluid. This improves the classical solution of Clebsch (1859) [4]. From about a half century ago, there was a fundamental question how rotational component can be formulated in the variational framework for an ideal compressible fluid. The present improved solution is an answer to the *long-standing* question.

Total kinetic energy and momentum are global integrals of $\frac{1}{2}\rho \langle \boldsymbol{v}, \boldsymbol{v} \rangle$ and $\rho \boldsymbol{v}$ respectively that characterize the flow field globally. There is another important global integral (a conservative integral over the

whole space), which is the *helicity* \mathcal{H} defined by

$$\mathcal{H} \equiv \int \langle \boldsymbol{v}, \boldsymbol{\omega} \rangle \, \mathrm{d}^3 \boldsymbol{x} \,, \tag{2}$$

where $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$ is the vorticity.

Associated with two symmetries (translation and space-rotation) of the fluid flow, there are two gauge fields $E = (v \cdot \nabla)v$ and $H = \nabla \times v$, which do not exist in the system of discrete masses, characterized by the same symmetries [2, 9]. One can show that those are analogous to the electric field and magnetic field in the electromagnetism, and fluid Maxwell equations can be formulated for them [3, 8]. This will be described in the section 4.

In the next section 2, basic properties of flow fields are summarized. In §3, first, the action principle for fluid flows is summarized for two different approaches of Lagrangian and Eulerian point of view. In the former case the variations are taken for the particle positions, while in the latter case, the variations are taken for all the field variables independently. The Euler-Lagrange equation for the former Lagrangian variation results in the Euler's equation of motion. On the other hand, a traditional approach of the Eulerian variation leads to a system of representations equivalent to the classical Clebsch solution [4]. In this solution, the vorticity has a special form such that the helicity vanishes. In a particular case of isentropic fluid in which the entropy s is a constant, the flow field thus obtained becomes *irrotational*, as mentioned above.

It is well-known that, even in such an isentropic fluid, the fluid flow can support rotational velocity fields. In fact, Euler [5] showed already in 1755 that his equation of motion can describe rotational flows. In the second half of §3, a new variational formulation is proposed and a new representation of compressible rotational flows of an ideal fluid is deduced.

2 Basic properties of flow fields

2.1 Equations of fluid mechanics

Equation of motion of an ideal fluid is given by the Euler's equation of motion:

$$\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla) \boldsymbol{v} = -\frac{1}{\rho} \nabla p.$$
 (3)

This is supplemented by the followings:

$$\partial_t \rho + \boldsymbol{v} \cdot \nabla \rho + \rho \,\nabla \cdot \boldsymbol{v} = 0, \qquad (4)$$

$$\partial_t s + \boldsymbol{v} \cdot \nabla s = 0, \qquad (5)$$

$$\partial_t \boldsymbol{\omega} + \nabla \times (\boldsymbol{\omega} \times \boldsymbol{v}) = 0, \qquad (6)$$

where ρ is the fluid density, s the entropy per unit mass, p the pressure. The equation (4) is the continuity equation, and (6) is the vorticity equation, while

(5) is the entropy equation, stating that each fluid particle keeps its initial entropy (*i.e.* adiabatic).

If initial entropy field is uniform with a constant value s_0 , the fluid keeps the *isentropic* state $s = s_0$ at any later time and everywhere. In this case, we have $(1/\rho)\nabla p = \nabla h$ by the thermodynamics where h is the enthalpy per unit mass.¹ In isentropic flows, an enthalpy variation Δh and a density variation $\Delta \rho$ are related by

$$\Delta h = \frac{1}{\rho} \Delta p = \frac{a^2}{\rho} \Delta \rho, \tag{7}$$

where $\Delta p = a^2 \Delta \rho$ and $a^2 = (\partial p/\partial \rho)_s$. The form $(\partial p/\partial \rho)_s$ denotes the derivative with *s* fixed, and $a = \{(\partial p/\partial \rho)_s\}^{1/2}$ is the *sound speed*. From the above, we have $\partial_t \rho = (\rho/a^2)\partial_t h$ and $\nabla \rho = (\rho/a^2)\nabla h$. Therefore, the equation (4) is transformed to $(\rho/a^2)(\partial_t h + \boldsymbol{v} \cdot \nabla h + a^2 \nabla \cdot \boldsymbol{v}) = 0$. Thus, the fluid equations (3) and (4) reduce to the followings:

$$\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla) \boldsymbol{v} + \nabla h = 0, \qquad (8)$$

$$\partial_t h + \boldsymbol{v} \cdot \nabla h + a^2 \nabla \cdot \boldsymbol{v} = 0. \tag{9}$$

2.2 Equations of electromagnetism

In electromagnetism, two vector fields E^{em} and H^{em} can be defined in terms of a vector potential A^{em} and a scalar potential ϕ^{em} by

$$\boldsymbol{E}^{\text{em}} = -c^{-1}\partial_t \boldsymbol{A}^{\text{em}} - \nabla \phi^{\text{em}}, \quad \boldsymbol{H}^{\text{em}} = \nabla \times \boldsymbol{A}^{\text{em}}.$$
(10)

Using these definitions,² Maxwell's equations are satisfied if the two fields A^{em} and ϕ^{em} satisfy the following equations ([6], Chap.8):

$$\partial_t \phi^{\text{em}} + c \,\nabla \cdot \boldsymbol{A}^{\text{em}} = 0, \quad \text{(Lorentz condition)}, (11)$$

$$\left(\partial_t^2 - c^2 \nabla^2\right) \phi^{\rm em} = c^2 q^{\rm e},\tag{12}$$

$$(\partial_t^2 - c^2 \nabla^2) \boldsymbol{A}^{\text{em}} = c^2 \boldsymbol{J}^{\text{e}}, \qquad (13)$$

where c is the light velocity, q^{e} the charge density, and J^{e} the current density vector.

2.3 Wave property and gauge invariance

Linearizing (8) by neglecting $(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$, and linearizing (9) by neglecting $\boldsymbol{v} \cdot \nabla h$ and replacing *a* with a constant value a_0 , we have

$$\partial_t \boldsymbol{v} + \nabla h = 0, \quad \partial_t h + a_0^2 \, \nabla \cdot \boldsymbol{v} = 0.$$
 (14)

Eliminating v from the two equations, we obtain the wave equation $(\partial_t^2 - a_0^2 \nabla^2)h = 0$ for sound waves.

¹From the thermodynamics, $dh = (1/\rho) dp + T ds$ where T is the temperature. If ds = 0, we have $dh = (1/\rho) dp$.

²Vector potential and scalar potential were used already by Maxwell (1873), and the equation of electromagnetic wave was derived by using the vector potential.

Using it, we obtain the wave equation for v as well. Thus, we have

$$(\partial_t^2 - a_0^2 \nabla^2)h = 0, \quad (\partial_t^2 - a_0^2 \nabla^2) \, \boldsymbol{v} = 0.$$
 (15)

It is remarkable that we have a close analogy between the two systems of fluid and electromagnetism. In vacuum space where $q^e = 0$, $J^e = 0$, the wave equations (??) reduce to

$$(\partial_t^2 - c^2 \nabla^2) \phi^{\text{em}} = 0, \quad (\partial_t^2 - c^2 \nabla^2) \mathbf{A}^{\text{em}} = 0.$$
(16)

It is seen that c (light speed) $\Leftrightarrow a_0$ (sound speed). Notable feature of the sund wave equations (15) is *non-dispersive*. Namely, the dispersion relation for waves of wave number k and frequency ω is given by $\omega^2 = a_0^2 k^2$, and the phase speed ω/k is equal to a_0 independent of the wave length $2\pi/k$. The same is true for the equation (16) of the electromagnetic wave. Note that this is closely related to the system of Maxwell equations.

In addition, the second equation of (14), obtained from the continuity equation by linearization, is analogous to the *Lorentz condition* (11) by the correspondence,

$$(c, \mathbf{A}^{\text{em}}, \phi^{\text{em}}) \iff (a_0, a_0 \boldsymbol{v}, h).$$

This implies possibility of formulation of *fluid* Maxwell equations for which the vector potential is played by v (except for the coefficient a_0 , abbreviated in the present formulation for simplicity) and the scalar potential played by h. According to this finding, let us propose two vector fields defined by

$$\boldsymbol{E} = -\partial_t \boldsymbol{v} - \nabla h, \qquad \boldsymbol{H} = \nabla \times \boldsymbol{v}.$$
 (17)

Consider the following transformations: $v' = v + \nabla f$, $h' = h - \partial_t f$. The vector fields E' and H' defined by v' and h' are unchanged, namely we have E' = E and H' = H. Thus in fluid flows, there exists the same gauge invariance as that in the electromagnetism. [6]

3 Variational principle

In the variational principle of field theory, variations are taken for all the field variables independently. This is called the *Eulerian* variation in fluid mechanics [7]. Let us define a Lagrangian density $\Lambda_{\rm Eul}$ by

$$\Lambda_{\text{Eul}}(\boldsymbol{v},\rho,s) = \frac{1}{2} \rho \langle \boldsymbol{v}, \boldsymbol{v} \rangle - \rho \,\epsilon(\rho,s), \qquad (18)$$

[2, 7], where $\epsilon(\rho, s)$ is the internal energy per unit mass depending on ρ and the entropy s. We take variations of the variables v, ρ and s, by assuming all the

variations being independent. On the other hand, in the *Lagrangian* variation, variations are taken with respect to the particle position $\mathbf{X}(\mathbf{a},\tau)$ with \mathbf{a} fixed, where the Lagrangian parameters $\mathbf{a} = (a_1, a_2, a_3)$ are defined by the initial particle position $\mathbf{a} = \mathbf{X}(\mathbf{a}, 0)$ and τ is the time used in combination with \mathbf{a} .

3.1 Lagrangian variation

The Lagrangian density Λ_{Lag} in this case is defined by

$$\Lambda_{\text{Lag}}\left(\boldsymbol{X}, \partial_{\boldsymbol{a}}\boldsymbol{X}\right) = \frac{1}{2}\rho\left(\partial_{\tau}\boldsymbol{X}\right)^{2} - \rho\epsilon(\boldsymbol{X}, \partial_{\boldsymbol{a}}\boldsymbol{X}), \quad (19)$$

where $\partial_{\tau} \mathbf{X} \equiv \partial \mathbf{X} / \partial \tau = \mathbf{v}$ is the particle velocity, and $(\partial_{\mathbf{a}} \mathbf{X})_{kj} = \partial X^k / \partial a_j$, written also as X_j^k . Time derivative is denoted by $X_0^k = \partial X^k / \partial \tau$. The mass of the fluid particle \mathbf{a} of volume dV is given by $m = \rho \, dV$. Considering the following variations,

$$X^k \to X^k + \delta X^k, \quad X^k_\mu \to X^k_\mu + \partial(\delta X^k) / \partial a_\mu.$$

for k = 1, 2, 3; $\mu = 0, \dots, 3$, and substituting these into (19), we obtain the Euler-Lagrange equation,

$$\frac{\partial}{\partial a_{\mu}} \left(\frac{\partial \Lambda}{\partial X_{\mu}^{k}} \right) - \frac{\partial \Lambda}{\partial X^{k}} = 0,$$

from the variational principle. This results in the equation of motion, $X_{\tau\tau}^k = -(1/\rho) \partial_k p$, for k = 1, 2, 3 (see [3, §7.4]). This transforms to the Euler's equation of motion by replacing $X_{\tau\tau}^k = \partial_{\tau}^2 X^k$ with $\partial_t v^k + (\boldsymbol{v} \cdot \nabla) v^k$, according to Leonhard Euler (1755) [5]. Thus, the Euler's equation of motion (3) has been derived by the Lagrangian variation.

3.2 Eulerian variation

For the Eulerian variation, it is proposed that the Lagrangian (18) is supplemented by additional terms associated with conservations of mass, entropy and vorticity. Thus the total Lagrangian L_* is defined by

$$L_{*} = \int_{V} \Lambda(\boldsymbol{v}, \rho, s, \phi, \psi, \boldsymbol{A}, \boldsymbol{\Omega}) d^{3}\boldsymbol{x}, \quad (20)$$

$$\Lambda = \frac{1}{2} \rho \langle \boldsymbol{v}, \boldsymbol{v} \rangle - \rho \epsilon(\rho, s)$$

$$-\rho D_{t} \phi - \rho s D_{t} \psi - \langle \mathcal{L}_{t}^{*}[\boldsymbol{A}], \boldsymbol{\Omega} \rangle, \quad (21)$$

where the last term $\langle \mathcal{L}_t^*[\boldsymbol{A}], \boldsymbol{\Omega} \rangle$ is a new term introduced in the present formuation from the view point of gauge theory. All the terms of the Lagrangian density Λ have the forms, which do not violate the symmetry consideration of gauge theory (see Appendices A and B, or [2, 3]). The domain V is a volume in the *x*-space chosen arbitrarily, Λ is the Lagrangian density, $\phi(\boldsymbol{x}, t)$ and $\psi(\boldsymbol{x}, t)$ are scalar potentials associated with mass and entropy. The vectors $\boldsymbol{\Omega}$ and \boldsymbol{A} are potentials of tangent vector and cotangent vector respectively.³ The vector Ω is assumed to be solenoidal:

$$\partial_k \Omega^k = 0. \tag{22}$$

An action integral is defined by

$$J = \int_{t_1}^{t_2} L_* \mathrm{d}t = \int_{V \otimes I_t} \Lambda(\boldsymbol{v}, \rho, s, \phi, \psi, \boldsymbol{A}, \boldsymbol{\Omega}) \, \mathrm{d}^4 x, \quad (23)$$

where $d^4x = dt d^3x$, and the time interval $I_t = [t_1, t_2]$ is chosen arbitrarily.

The *action principle* is defined by

$$\delta J = \int_{V \otimes I_t} \delta \Lambda(\boldsymbol{v}, \rho, \boldsymbol{s}, \phi, \psi, \boldsymbol{A}, \boldsymbol{\Omega}) \, \mathrm{d}^4 \boldsymbol{x} = 0, \quad (24)$$

for its variation δJ with respect to arbitrary variations of the variables $\boldsymbol{v}, \rho, s, \phi, \psi, \boldsymbol{A}$, and $\boldsymbol{\Omega}$, where all the variations are assumed to be independent, and also to vanish on the boundary surface *S* enclosing the integration domain $V \otimes I_t$. Substituting the varied variables $\boldsymbol{v} + \delta \boldsymbol{v}, \rho + \delta \rho, s + \delta s, \phi + \delta \phi, \psi + \delta \psi,$ $\boldsymbol{A} + \delta \boldsymbol{A}, \text{ and } \boldsymbol{\Omega} + \delta \boldsymbol{\Omega} \text{ into } \Lambda[v_i, \rho, s, \phi, \psi, \boldsymbol{A}, \boldsymbol{\Omega}]$ and writing its variation as $\delta \Lambda$, we obtain

$$\delta \Lambda = \delta v_i \left[\rho \left(v_i - \partial_i \phi - s \, \partial_i \psi \right) - \Omega_k \, \partial_i A_k \right. \\ \left. + \Omega_k \, \partial_k A_i \right] - \partial_k \left(\Omega_k A_i \, \delta v_i \right)$$
(25)

$$+ \delta\rho \left(\frac{1}{2}u^{2} - h - \mathbf{D}_{t}\phi - s \mathbf{D}_{t}\psi\right) - \delta s \rho \left(\mathbf{D}_{t}\psi + T\right) \\ + \delta\phi \left(\partial_{t}\rho + \nabla \cdot \left(\rho v\right)\right) - \partial_{t}(\rho \delta\phi) - \nabla \cdot \left(\rho v \,\delta\phi\right) \\ + \delta\psi \left(\partial_{t}(\rho s) + \nabla \cdot \left(\rho s v\right)\right) - \partial_{t}(\rho s \,\delta\psi) \\ - \nabla \cdot \left(\rho s v \,\delta\psi\right). \\ - \langle \mathcal{L}_{t}^{*}[\boldsymbol{A}], \,\delta\boldsymbol{\Omega} \rangle + \left\langle \delta\boldsymbol{A}, \left(\mathcal{L}_{t}[\boldsymbol{\Omega}] + \boldsymbol{\Omega} \,\partial_{k}v^{k}\right) \right\rangle.$$

where $h = \epsilon + p/\rho$ is the specific enthalpy, and standard relations of thermodynamics are used.⁴ As usual, all the surface integrals (obtained with integration by parts) vanish by the assumed boundary conditions. Thus, the variational principle $\delta J = \int \delta \Lambda d^4 x = 0$ for independent variations of δv_i , $\delta \rho$, δs , etc. results in the followings:

$$\delta v_i : \rho \left(v_i - \partial_i \phi - s \, \partial_i \psi \right) - \Omega_k \, \partial_i A_k + \Omega_k \, \partial_k A_i = 0 \,, \quad (26)$$

$$\delta \rho$$
 : $\frac{1}{2}v^2 - h - D_t \phi - s D_t \psi = 0$, (27)

$$\delta s : \mathbf{D}_t \psi + T = 0, \qquad (28)$$

³Scalar product between two tangent vectors $\boldsymbol{a} = (a^i)$ and $\boldsymbol{b} = (b^i)$ is denoted in this paper by four ways, $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$, $\boldsymbol{a} \cdot \boldsymbol{b}$, $\delta_{ij}a^ib^j$, or a_ib^i , where $a_i(=\delta_{ij}a^j)$ is a cotangent vector (derived from the tangent vector a^i).

 ${}^{4}(\partial\epsilon/\partial\rho)_{s} = p/\rho^{2}, \ (\partial/\partial\rho)_{s}(\rho \epsilon) = \epsilon + \rho (\partial\epsilon/\partial\rho)_{s} = \epsilon + p/\rho = h, \text{ and } (\partial\epsilon/\partial s)_{\rho} = T.$ Then we have $d\epsilon = (p/\rho^{2}) d\rho + Tds$ and $dh = (1/\rho) dp + Tds$.

$$\delta\phi \quad : \quad \partial_t \rho + \nabla \cdot (\rho \boldsymbol{v}) = 0 \,, \tag{29}$$

$$\delta\psi : \quad \partial_t(\rho s) + \nabla \cdot (\rho s \boldsymbol{v}) = 0, \qquad (30)$$

$$\delta \mathbf{\Omega} \quad : \quad \mathcal{L}_t^*[\mathbf{A}] = 0. \tag{31}$$

$$\delta \boldsymbol{A} : \mathcal{L}_t[\boldsymbol{\Omega}] + \boldsymbol{\Omega} \,\partial_k v^k = \partial_t \boldsymbol{\Omega} + \nabla \times (\boldsymbol{\Omega} \times \boldsymbol{v}) = 0, \quad (32)$$

The equation (26) gives a *new expression* of v:

$$\boldsymbol{v} = \nabla \phi + s \,\nabla \psi + \frac{1}{\rho} \,\boldsymbol{w} \,, \tag{33}$$

where $\boldsymbol{w} = (w_i)$ is given by

$$\boldsymbol{w} = \Omega^k \nabla A_k - (\boldsymbol{\Omega} \cdot \nabla) \boldsymbol{A} = \boldsymbol{\Omega} \times (\nabla \times \boldsymbol{A}), \quad (34)$$

$$w_i = \Omega^k C_{ik}, \quad C_{ik} = \partial_i A_k - \partial_k A_i.$$
 (35)

Note that we have the equality,

$$\Omega^k C_{ik} = [\boldsymbol{\Omega} \times (\nabla \times \boldsymbol{A})]_i.$$

The vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$ is given by

$$\boldsymbol{\omega} = \nabla s \times \nabla \psi + \frac{1}{\rho} \nabla \times \boldsymbol{w} - \frac{1}{\rho^2} \nabla \rho \times \boldsymbol{w}.$$
 (36)

The second and third terms express non-vanishing vorticity *even* in an isentropic fluid of uniform s. Defining $B = \nabla \times A$, we have

$$\nabla \times \boldsymbol{w} = (\boldsymbol{B} \cdot \nabla) \boldsymbol{\Omega} - (\boldsymbol{\Omega} \cdot \nabla) \boldsymbol{B}$$
 (37)

$$= \nabla \Omega_k \times \nabla A_k - \nabla \Omega_k \times \partial_k \boldsymbol{A} - (\boldsymbol{\Omega} \cdot \nabla) \boldsymbol{B}.$$
(38)

Thus, the present formulation yields the rotational component naturally by the variational principle.

From the variations of $\delta \phi$ and $\delta \psi$, we obtain

$$\delta \phi : \quad \partial_t \rho + \nabla \cdot (\rho \boldsymbol{v}) = 0, \quad (39)$$

$$\delta \psi : \quad \partial_t (\rho s) + \nabla \cdot (\rho s \boldsymbol{v}) = 0.$$

Using (39), the second reduces to the *adiabatic* equation:

$$\partial_t s + \boldsymbol{v} \cdot \nabla s = \mathbf{D}_t s = 0. \tag{40}$$

Thus, we obtain the continuity equation (39) and entropy equation (40) from the action principle. The equation (39) can be rewritten as

$$\partial_t \rho + \boldsymbol{v} \cdot \nabla \rho + \rho \nabla \cdot \boldsymbol{v} = \mathsf{D}_t \rho + \rho \nabla \cdot \boldsymbol{v} = 0. \quad (41)$$

3.3 Traditional variation

In the traditional Eulerian variation, the Lagrangian density Λ does not include the last term $\langle \mathcal{L}_t^*[\mathbf{A}], \mathbf{\Omega} \rangle$ of (21). As a result, the velocity v does not include the last term $(1/\rho)w$ of the velocity (33). Then, the vorticity ω_* is given by

$$\boldsymbol{\omega}_* = \nabla \times \boldsymbol{v} = \nabla \boldsymbol{s} \times \nabla \boldsymbol{\psi}, \tag{42}$$

which vanishes in an isentropic fluid of uniform s. The scalar product $\omega_* \cdot v$ is given by

$$\boldsymbol{\omega}_* \cdot \boldsymbol{v} = (\nabla s \times \nabla \psi) \cdot (\nabla \phi + s \nabla \psi) = \boldsymbol{\omega}_* \cdot \nabla \phi = \nabla \cdot [\phi \, \boldsymbol{\omega}_*].$$

The helicity \mathcal{H} is defined by the integral (2). Assuming that ω_* vanishes out of V, we have

$$\mathcal{H} \equiv \int_{V} \boldsymbol{\omega}_{*} \cdot \boldsymbol{v} \mathrm{d}^{3} \boldsymbol{x} = \int_{V} \nabla \cdot [\phi \, \boldsymbol{\omega}_{*}] \mathrm{d}^{3} \boldsymbol{x} = 0. \quad (43)$$

However, for the present velocity fields (33), the \mathcal{H} does not vanish in general, demonstrating that it is a new representation. The helicity \mathcal{H} is a measure of linkage and knottedness of vortex lines.

3.4 Clebsch solution

The results of traditional variational principle without the last term $\langle \mathcal{L}_t^*[\mathbf{A}], \mathbf{\Omega} \rangle$ of (21) are summarized as

$$\boldsymbol{v} = \nabla \phi + s \, \nabla \psi, \tag{44}$$

$$\frac{1}{2}v^2 + h + \partial_t \phi + s \,\partial_t \psi = 0, \qquad (45)$$

$$D_t s = 0, \qquad D_t \psi + T = 0.$$
 (46)

The velocity field (44) is equivalent to the classical Clebsch solution [3]. In fact, using (44) and (42), and using a vector identity⁵, we have

$$\boldsymbol{\omega} \times \boldsymbol{v} = (\boldsymbol{v} \cdot \nabla s) \nabla \psi - (\boldsymbol{v} \cdot \nabla \psi) \nabla s, \partial_t \boldsymbol{v} = \nabla \partial_t \phi + \partial_t s \nabla \psi + s \nabla \partial_t \psi.$$

Adding the last two equations, we obtain

$$\partial_t \boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{v} = \nabla (\partial_t \phi + s \, \partial_t \psi) + (\mathbf{D}_t s) \, \nabla \psi - (\mathbf{D}_t \psi) \, \nabla s,$$

Last two terms become $T\nabla s$ due to (46). Thus, by using (45), the following is satisfied:

$$\partial_t \boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{v} = -\nabla(\frac{1}{2}v^2 + h) + T\nabla s.$$
 (47)

Owing to the relation $dh = (1/\rho) dp + T ds$ of the footnote 1, this reduces to the Euler's equation (3) owing to the following vector identity,

$$(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = (\nabla\times\boldsymbol{v})\times\boldsymbol{v} + \nabla(\frac{1}{2}v^2).$$
 (48)

$${}^{5}(\nabla s \times \nabla \psi) \times \boldsymbol{v} = -(\boldsymbol{v} \cdot \nabla \psi) \, \nabla s + (\boldsymbol{v} \cdot \nabla s) \, \nabla \psi.$$

3.5 Terms of total time derivative

Using the results of previous subsections, one can show that both of the two terms of the Lagrangian Λd^3x of (21), *i.e.* $\rho D_t \phi d^3x$ and $\rho s D_t \psi d^3x$, are represented in the form of Lie-derivative.

Lie-derivative $\mathcal{L}_t X$ takes different forms depending on the object X of operation [3]. Here we show its form when X is a scalar field ϕ (a *zero-form*), or d^3x a volume *three-form*:

$$\mathcal{L}_t[\phi] \equiv \partial_t \phi + v^k \partial_k \phi = \mathbf{D}_t \phi, \qquad (49)$$

$$\mathcal{L}_t[\mathbf{d}^3 \boldsymbol{x}] \equiv (\partial_k v^k) \, \mathbf{d}^3 \boldsymbol{x}. \tag{50}$$

(See (55) and (56) for the Lie-derivatives of a tangent vector, or a cotangent vector.) Rate of change of mass during the motion is given by

$$\begin{aligned} \mathcal{L}_t[\rho\,\mathrm{d}^3\boldsymbol{x}] &= \mathcal{L}_t[\rho]\,\mathrm{d}^3\boldsymbol{x} + \rho\,\mathcal{L}_t[\mathrm{d}^3\boldsymbol{x}] \\ &= (\partial_t\rho + v^k\partial_k\rho + \rho\partial_kv^k)\,\mathrm{d}^3\boldsymbol{x} = 0, \end{aligned}$$

by (41). Therefore we obtain

$$\mathcal{L}_t[\phi \rho \,\mathrm{d}^3 \boldsymbol{x}] = (\mathrm{D}_t \phi) \,\rho \,\mathrm{d}^3 \boldsymbol{x} + \phi \,\mathcal{L}_t[\rho \,\mathrm{d}^3 \boldsymbol{x}] = (\mathrm{D}_t \phi) \,\rho \,\mathrm{d}^3 \boldsymbol{x}.$$
(51)

Similarly, using $D_t s = 0$, we obtain

$$\mathcal{L}_t \left[\psi \, s \, \rho \, \mathrm{d}^3 \boldsymbol{x} \right] = (\mathrm{D}_t \psi) \, s \rho \, \mathrm{d}^3 \boldsymbol{x}. \tag{52}$$

Thus it is found that the two terms of (21) can be written in the following form of Lie-derivative:

$$(\rho \mathbf{D}_t \phi + \rho s \mathbf{D}_t \psi) \mathbf{d}^3 \boldsymbol{x} = \mathcal{L}_t \Big[(\phi \, \rho + \psi \, s \, \rho) \, \mathbf{d}^3 \boldsymbol{x} \Big].$$
(53)

This can be rewritten as $(\partial_t \Phi + \partial_k (v^k \Phi)) d^3 x$, where $\Phi = \phi \rho + \psi s \rho$. In the action J of (23), the terms $\partial_t \Phi + \partial_k (v^k \Phi)$ are integrated over the domain $V \otimes I_t$, which are transformed to surface integrals over S bounding it. Hence they do not influence the Euler-Lagrange equation derived from the action principle of the Lagrange variation.

The last term $\langle \mathcal{L}_t^*[\mathbf{A}], \mathbf{\Omega} \rangle$ of (21) is also written in the form of Lie-derivative, by assuming that the vector field $\mathbf{\Omega}$ satisfies the equation of frozen field,

$$\partial_t \mathbf{\Omega} + \nabla \times (\mathbf{\Omega} \times \boldsymbol{v}) = \mathcal{L}_t[\mathbf{\Omega}] + \mathbf{\Omega} \,\partial_k v^k = 0.$$
 (54)

Lie-derivatives of a tangent vector $\mathbf{\Omega} = (\Omega^i)$ and a cotangent vector $\mathbf{A} = (A_i)$ (a *one-form*) are given by

$$(\mathcal{L}_t[\mathbf{\Omega}])^i \equiv \partial_t \Omega^i + v^k \partial_k \Omega^i - \Omega^k \partial_k v^i,$$
 (55)
$$(\mathcal{L}_t^*[\mathbf{A}])_i \equiv \partial_t A_i + v^k \partial_k A_i + A_k \partial_i v^k.$$
 (56)

By using these definitions, we have

$$\langle \mathcal{L}_t^*[\boldsymbol{A}], \boldsymbol{\Omega} \rangle \, \mathrm{d}^3 \boldsymbol{x} = \mathcal{L}_t \Big[\langle \boldsymbol{A}, \, \boldsymbol{\Omega} \rangle \, \mathrm{d}^3 \boldsymbol{x} \Big]$$
 (57)

owing to the equation (54). Integration of (57) over $V \otimes I_t$ is also transformed to surface integrals over S. In fact, the equation (54) is required by the action principle, as given by (32).

Thus, it has been shown that the three terms of (21) added to $\Lambda_{\rm Eul}$ of (18) are all transformed to surface integrals over the boundary S. Hence those do not influence the Euler-Lagrange equation derived from the action principle in the Lagrange variation, although they are added to the Lagrangian in the Eulerian variation.

3.6 Euler's equation of motion is satisfied

Last step is to verify that the set of equations derived in the section §3.2 *in fact* satisfies the Euler's equation of motion. This is carried out for the flow field described by (26), (27) and (28). Applying the covariant derivative $D_t = \partial_t + \mathbf{v} \cdot \nabla$ to \mathbf{v} of (33), we have

$$\mathbf{D}_{t}[\boldsymbol{v}] = \mathbf{D}_{t}\nabla\phi + \mathbf{D}_{t}(s\,\nabla\psi) + \frac{1}{\rho}\,\mathbf{D}_{t}\boldsymbol{w} - \frac{1}{\rho^{2}}\,(\mathbf{D}_{t}\rho)\,\boldsymbol{w},$$
(58)

where, from (41),

$$\mathbf{D}_t \rho = -\rho \left(\partial_k v^k \right). \tag{59}$$

The first term can be rewritten as

$$\mathbf{D}_t(\nabla \phi) = \nabla(\mathbf{D}_t \phi) - \partial_k \phi \, \nabla v^k. \tag{60}$$

Using (40) and (28), the second term of (58) is

$$D_t(s \nabla \psi) = s \nabla (D_t \psi) - s \partial_k \psi \nabla v^k$$

= $-s \nabla T - s \partial_k \psi \nabla v^k$. (61)

By using (35), the third term is

$$\mathbf{D}_t w^i = \mathbf{D}_t(\Omega^k) C_{ik} + \Omega^k \mathbf{D}_t(C_{ik}).$$
 (62)

By (32) and (55), we have

$$\mathbf{D}_t(\Omega^k) = \Omega^l \,\partial_l v^k - \Omega^k \,\partial_l v^l \,,$$

while for $D_t(C_{ik})$, by using (35), we have

$$\begin{aligned} \mathbf{D}_t C_{ik} &= \partial_i (\mathbf{D}_t A_k) - \partial_k (\mathbf{D}_t A_i) \\ &- (\partial_i v^l) \, \partial_l A_k + (\partial_k v^l) \, \partial_l A_i \,. \end{aligned}$$

Substituting these two into (62),

$$\begin{aligned} \mathbf{D}_{t}w_{i} &= \Omega^{l} \,\partial_{l}v^{k} \left(\partial_{i}A_{k} - \partial_{k}A_{i}\right) \\ &- \Omega^{k} \,\partial_{l}v^{l} \left(\partial_{i}A_{k} - \partial_{k}A_{i}\right) + \Omega^{k} \left(\partial_{i}(\mathbf{D}_{t}A_{k}) \right. \\ &\left. - \partial_{k}(\mathbf{D}_{t}A_{i}) - \left(\partial_{i}v^{l}\right) \partial_{l}A_{k} + \left(\partial_{k}v^{l}\right) \partial_{l}A_{i} \right). \end{aligned}$$

The right hand side simplifies greatly by cancellation, and we finally obtain

$$D_{t}\boldsymbol{w} = -w_{k} \nabla v^{k} - (\partial_{l}v^{l}) w_{i}, \qquad (63)$$
$$w_{k} = \Omega^{l} C_{kl}.$$

Substituting (59), (60), (61) and (63) into (58),

$$\begin{aligned} \mathbf{D}_t \boldsymbol{v} &= \nabla (\mathbf{D}_t \phi) - (\partial_k \phi + s \partial_k \psi + \frac{1}{\rho} w_k) \nabla v^k - s \nabla T \\ &= \nabla (\mathbf{D}_t \phi) - v_k \nabla v^k - s \nabla T \\ &= \nabla (\mathbf{D}_t \phi - \frac{1}{2} v^2 - sT) + T \nabla s. \end{aligned}$$

Using (27) and (28), this reduces to the Euler's equation of motion (3):

$$\mathbf{D}_t \boldsymbol{v} = -\nabla h + T \,\nabla s = -\frac{1}{\rho} \,\nabla p \,. \tag{64}$$

since $dh = (1/\rho) dp + T ds$. Thus, it is found that the present *improved* variation which takes account of the new term $\langle \mathcal{L}_t^*[\mathbf{A}], \mathbf{\Omega} \rangle$ in (21) leads to a *new* result, *i.e.* Euler's equation of motion is satisfied by the new set of (26), (27) and (28), representing *rotational compressible* flow field of an ideal fluid.

In the case of isentropic flows of uniform s, the equation (64) reduces to

$$\partial_t \boldsymbol{v} + (\nabla \times \boldsymbol{v}) \times \boldsymbol{v} + \nabla(\frac{1}{2}v^2) = -\nabla h,$$
 (65)

by replace $D_t v$ with $(\partial_t + v \cdot \nabla) v$ first and then using (48) next. Taking curl of (65) and setting $\nabla \times v = \omega$, we obtain the vorticity equaiton (6),

$$\partial_t \boldsymbol{\omega} + \nabla \times (\boldsymbol{\omega} \times \boldsymbol{v}) = 0.$$
 (66)

4 Fluid Maxwell Equations

The analysis of §2.3 implied possibility of formulation of *fluid Maxwell equations*. In this section, its system of equations is presented, starting from the definition of two vector fields E and H (analogous to (10)) given by

$$\boldsymbol{E} = -\partial_t \boldsymbol{v} - \nabla h, \qquad \boldsymbol{H} = \nabla \times \boldsymbol{v}, \qquad (67)$$

Fluid Maxwell Equations can be derived from the fluid equations (8) and (9) as follows:

(A)
$$\nabla \cdot \boldsymbol{H} = 0,$$
 (68)

(B)
$$\nabla \times \boldsymbol{E} + \partial_t \boldsymbol{H} = 0,$$
 (69)

(C)
$$\nabla \cdot \boldsymbol{E} = q,$$
 (70)

(D)
$$a_0^2 \nabla \times \boldsymbol{H} - \partial_t \boldsymbol{E} = \boldsymbol{J},$$
 (71)

$$q \equiv \nabla \cdot [(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}], \tag{72}$$

$$\boldsymbol{J} \equiv \partial_t^2 \boldsymbol{v} + \nabla \partial_t h + a_0^2 \nabla \times (\nabla \times \boldsymbol{v}), \quad (73)$$

[3, 8, 9], where a_0 is a constant (the sound speed in undisturbed state). From the calculus $\partial_t(C) + \text{div}(D)$, we have the *charge conservation*: $\partial_t q + \text{div} J = 0$.

Using (8) and the definition $E = -\partial_t v - \nabla h$, the *fluid-electric* field E is given by

$$\boldsymbol{E} = (\boldsymbol{v} \cdot \nabla) \boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{v} + \nabla(\frac{1}{2} v^2).$$
(74)

where (48) is used.

4.1 Derivation

Derivation of the fluid Maxwell equations (68) \sim (71) is carried out as follows.

- (*a*) Equation (A) is deduced immediately from the definition of $H = \nabla \times v = \omega$.
- (b) Equation (B) is an identity obtained from the definition (67). Moreover, if the expression (74) is substituted to *E* and ω to *H*, then the equation (B) reduces to the vorticity equation (66).
- (*c*) Equation (C) is just div [Eq.(74)] with the *charge density q* defined by (72).
- (d) Equation (D) is derived as follows.

Applying ∂_t to $\boldsymbol{E} = -\partial_t \boldsymbol{v} - \nabla h$, we obtain

$$-\partial_t \boldsymbol{E} - \partial_t^2 \boldsymbol{v} = \nabla \partial_t h,$$

Adding the term $a_0^2 \nabla \times \boldsymbol{H} = a_0^2 \nabla \times (\nabla \times \boldsymbol{v})$ on both sides, this can be rearranged as follows:

$$a_0^2 \nabla \times \boldsymbol{H} - \partial_t \boldsymbol{E} = \boldsymbol{J}, \\ \boldsymbol{J} = \partial_t^2 \boldsymbol{v} + \nabla \partial_t h + a_0^2 \nabla \times (\nabla \times \boldsymbol{v}).$$

which is nothing but the equation (D).

The vector \boldsymbol{J} can be given another expression by using $\partial_t h = -(\boldsymbol{v} \cdot \nabla)h - a^2 \nabla \cdot \boldsymbol{v}$ from the continuity equation (9): This can be rewritten as

$$\begin{aligned} \boldsymbol{J} &= (\partial_t^2 - a_0^2 \nabla^2) \boldsymbol{v} + \boldsymbol{J}^*, \\ \boldsymbol{J}^* &= a_0^2 \, \nabla (\nabla \cdot \boldsymbol{v}) - \nabla \big(a^2 \, \nabla \cdot \boldsymbol{v} \big) - \nabla (\boldsymbol{v} \cdot \nabla \, h), \end{aligned}$$

where the following identity is used:

$$\nabla(\nabla \cdot \boldsymbol{v}) = \nabla \times (\nabla \times \boldsymbol{v}) + \nabla^2 \boldsymbol{v}, \quad (75)$$

5 Equation of Sound Wave

Suppose that a localized flow is generated at an initial instant in otherwise uniform state at rest, where undisturbed values of the pressure, density and enthalpy are respectively p_0 , ρ_0 and h_0 . Equation of sound wave is derived from the system of fluid Maxwell equations (A) ~ (D) as follows [8].

Differentiating Eq.(D) with respect to t, and eliminating $\partial_t H$ by using (B), we obtain

$$\partial_t^2 \boldsymbol{E} + a_0^2 \nabla \times (\nabla \times \boldsymbol{E}) = -\partial_t \boldsymbol{J}.$$
 (76)

The second term on the left can be rewritten by using the identity (75), with v replaced by E. Then the equation (76) reduces to

$$(\partial_t^2 - a_0^2 \nabla^2) (\boldsymbol{E} + \partial_t \boldsymbol{v}) = -a_0^2 \nabla (\nabla \cdot \boldsymbol{E}) - \partial_t \boldsymbol{J}^* (77) \boldsymbol{J}^* = \nabla ((a_0^2 - a^2) \nabla \cdot \boldsymbol{v}) - \nabla (\boldsymbol{v} \cdot \nabla h) \equiv a_0^2 \nabla \hat{Q}, \hat{Q} = (1 - \hat{a}^2) \nabla \cdot \boldsymbol{v} - a_0^{-2} (\boldsymbol{v} \cdot \nabla)h,$$
(78)

where $\hat{a} = a/a_0$. We have $\boldsymbol{E} + \partial_t \boldsymbol{v} = -\nabla h$ from (67), and also $\partial_t \boldsymbol{J}^* = a_0^2 \nabla \partial_t \hat{Q}$ from (78). Therefore, all the terms are of the form of gradient of scalar fields and (77) can be integrated spatially. Dividing (77) with $-a_0^2$ and integrating it, we obtain the following wave equation:

$$(a_0^{-2}\partial_t^2 - \nabla^2)\tilde{h} = S(\boldsymbol{x}, t),$$

$$S(\boldsymbol{x}, t) \equiv \nabla \cdot \boldsymbol{E} + \partial_t \hat{Q}, \qquad (79)$$

where $\hat{h} \equiv h - h_0 = (p - p_0)/\rho$. Thus, the vectorial form of wave equation (77) has been reduced to the single equation for a scalar field \tilde{h} (see the first of (15)). The term $S(\boldsymbol{x},t)$ is a source of the wave. Using (74), we obtain an explicit form of the first $\nabla \cdot \boldsymbol{E}$ of the source S as $\nabla \cdot \boldsymbol{E} =$ $\operatorname{div}(\boldsymbol{\omega} \times \boldsymbol{v}) + \nabla^2 \frac{1}{2} v^2$. The first term $\operatorname{div}(\boldsymbol{\omega} \times \boldsymbol{v})$ implies that the motion of $\boldsymbol{\omega}$ generates sound waves. This is the source term of the *Vortex sound* (Kambe 2010c), and contribution from the second term $\nabla^2 \frac{1}{2} v^2$ vanishes in an ideal fluid in which total kinetic energy $\int \frac{1}{2} v^2 d^3 \boldsymbol{x}$ is conserved. Mach number of the source flow is defined by $M = |\boldsymbol{v}|/a_0$, then the second term $\partial_t \hat{Q}$ of S is $O(M^2)$, namely, higher order if M is small enough. [10]

6 Equation of motion of a test particle in a flow field

Analogy between fluid mechanics and electromagnetism is also found in the equation of motion of a test particle in a flow field as well [3, 8]. Suppose that a *test particle* of mass m is moving in a flow field v(x, t), which is *unsteady*, *rotational and compressible*. The size of the particle and its velocity are assumed to be so small that its influence on the background velocity field v(x, t) is negligible, namely the velocity field v is regarded as independent of the position and velocity of the particle.

The particle velocity is defined by u(t) relative to the fluid velocity v. Then, the total particle velocity is u+v. In this circumstance, the *i*-th component of total momentum P_i associated with the test particle moving in the flow field is expressed by the sum: $P_i = mu_i + m_{ik}u_k$,⁶ and the equation of motion of the particle is given by

⁶According to the hydrodynamic theory (*e.g.* Landau & Lifshitz (1987)) when a solid particle moves through the fluid (at rest), the fluid energy induced by the relative particle motion of velocity $\boldsymbol{u} = (u_i)$ is expressed in the form $\frac{1}{2} m_{ik} u_i u_k$ by using the *mass tensor* m_{ik} . Additional fluid momentum induced by the particle motion is given by $m_{ik} u_k$.

$$\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{P} = \boldsymbol{m} \boldsymbol{E} + \boldsymbol{m} \boldsymbol{u} \times \boldsymbol{H} - \boldsymbol{m} \nabla \phi_g, \quad (80)$$

(Kambe 2010b), where \boldsymbol{E} and \boldsymbol{H} take the same expressions as those of (67), and $\boldsymbol{P} = (P_i)$, $P_i = mu_i + m_{ik}u_k$, and $\phi_g = gz$. Obviously, the equation (80) is analogous to the equation of motion of a *charged* particle in an electromagnetism of electric field $\boldsymbol{E}^{\text{em}}$ and magnetic field $\boldsymbol{H}^{\text{em}}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\boldsymbol{v}_e) = e\,\boldsymbol{E}^{\mathrm{em}} + (e/c)\,\boldsymbol{v}_e \times \boldsymbol{H}^{\mathrm{em}} - m\nabla\Phi_g\,, \quad (81)$$

where v_e is the velocity of a charged particle, and c the light velocity. Rewriting the second term of (80) as $(m/a_0) u \times (a_0 H)$, and comparing the first two terms on the right of (80) and (81), it is found that there is correspondence: $e \Leftrightarrow m$, $E^{em} \Leftrightarrow E$, and $H^{em} \Leftrightarrow a_0 H$.

7 Conclusion

Following the scenrio of the gauge theory of physics, the fundamental framework of fluid mechanics has been reconsidered, and a new variational formulation is proposed for flows of an ideal fluid. This lead to a new representation of compressible rotational flows of an ideal fluid with an explicit expression of helicity. The system of new expressions satisfies the Euler's equation of motion.

Associated with two symmetries (translation and space-rotation) of flow field, there are two gauge fields, which do not exist in the system of discrete masses, and fluid Maxwell equations are formulated for the two gauge fields. Sound wave within the fluid is analogous to the electromagnetic wave, in the sense that phase speeds of both waves are independent of wave lengths, i.e. non-dispersive.

Thus, the present study of fluid flows on the basis of the gauge theory provides us a new playground on which one can develop new activity to study the dynamics of fluid flows.

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Appendix

A Background of the theory

This appendix reviews the background of the gauge theory and describes the scenario of the gauge principle in physical systems, from the Appendix A of [2].

A.1 Gauge invariances

In the theory of electromagnetism, it is well-known that there is an invariance under a *gauge transformation* of electromagnetic potentials consisting of a scalar potential ϕ and a vector potential A. The electric field E and magnetic field B are represented as $E = -\nabla \phi - (1/c)(\partial A/\partial t)$, and $B = \nabla \times A$, where c is the light velocity. The idea is that the fields E and B are unchanged by the following transformation: $(\phi, A) \rightarrow (\phi', A')$, where $\phi' = \phi - c^{-1}\partial_t f$, $A' = A + \nabla f$ with f(x, t) being an arbitrary differentiable scalar function of position x and time t.

Gauge invariance is also known in certain rotational flows of the Clebsch representation (Eckart 1960). Its velocity field is represented by (44) as

$$\boldsymbol{v} = \nabla \phi + s \,\nabla \psi,$$

where ϕ , ψ and s are scalar functions, and ψ satisfies the equation $D_t \psi = 0$. Let ϕ' , ψ' and s' be a second set giving the same velocity field v, which implies the following: $s \nabla \psi - s' \nabla \psi' = \nabla (\phi' - \phi)$. This relation will hold true by the relation $\phi' - \phi = F(\psi, \psi')$, if $s = \partial F/\partial \psi$ and $s' = -\partial F/\partial \psi'$. Therefore, s and ψ are determined only up to such a *contact* transformation, and ϕ transforms by the addition of the generating function $F(\psi, \psi')$. It can be shown that the two sets of triplet give the identical flow field. Thus, there exist some freedom in the expressions of physical fields in terms of potentials.

A.2 Related aspects in quantum mechanics and relativity theory

In quantum mechanics, Schrödinger's equation for a charged particle of mass m and electromagnetic fields are invariant with respect to a gauge transformation. This is as follows. In the *absence* of electromagnetic fields, Schrödinger's equation for a wave function ψ of a free particle m is written as

$$S_{\rm free}[\psi] \equiv i\hbar \,\partial_t \psi - \frac{1}{2m} \,p_k^2 \,\psi = 0,$$

where $p_k = -i\hbar \partial_k$ is the momentum operator ($\partial_k = \partial/\partial x^k$ for k = 1, 2, 3).

In the *presence* of electromagnetic fields, Schrödinger's equation for a particle with an electric charge e is obtained by the transformation:

$$\begin{aligned} \partial_t &\to \partial_t + (e/i\hbar)A_0, \\ \partial_k &\to \partial_k + (e/i\hbar c)A_k, \quad (k = 1, 2, 3), \end{aligned}$$
 (82)

where $(A_{\mu}) = (A_0, A_1, A_2, A_3) = (A_0, A^{\text{em}}) = (-\phi^{\text{em}}, A^{\text{em}})$ is a four-vector potential, from which the electromagnetic fields E^{em} and H^{em} are determined by (10) of §2.2 of main text. Thus, we obtain the equation with electromagnetic fields:

$$S_A[\psi] \equiv i\hbar \,\partial_t \psi - e\phi \,\psi \\ -\frac{1}{2m} \left[(-i\hbar)(\partial_k + \frac{e}{i\hbar c}A_k) \right]^2 \psi = 0.$$

A point in space-time is denoted by a four vector with upper indices, $(x^{\mu}) = (x^0, x^1, x^2, x^3)$ with $x^0 = ct$. Suppose that a wave function $\psi(x^{\mu})$ satisfies the equation $S_A[\psi] = 0$, and consider the following set of transformations of $\psi(x^{\mu})$ and A_{μ} :

$$\psi'(x^{\mu}) = \exp\left[i\,\alpha(x^{\mu})\right]\,\psi(x^{\mu}),$$

$$A_k \to A'_k = A_k + \partial_k\beta, \quad \phi \to \phi' = \phi - c^{-1}\partial_t\beta,$$

where $\alpha = (e/\hbar c)\beta$. Then, it is shown readily that the transformed function $\psi'(x^{\mu})$ satisfies the Schrödinger equation $S_{A'}[\psi'] = 0$. This is the *gauge invariance* of the system of an electric charge in electromagnetic fields. The system is said to have a gauge symmetry.

A.3 Brief scenario of gauge principle

In the gauge theory of particle physics, a free-particle Lagrangian $\Lambda_{\rm free}[\psi]$ is defined first for the wave function $\psi(x^{\mu})$ of a particle with an electric charge. Let us consider the following gauge transformation: $\psi \mapsto e^{i\alpha} \psi$. If $\Lambda_{\rm free}$ is invariant under the transformation when α is a constant, it is said that $\Lambda_{\rm free}$ has a *global* gauge invariance. In spite of this, it often happens that $\Lambda_{\rm free}$ is *not* invariant for a function $\alpha = \alpha(x)$, *i.e.* $\Lambda_{\rm free}$ is not gauge-invariant *locally.* In this circumstance, it is instrumental to introduce a new field in order to acquire local gauge invariance. If the new field (a gauge field) is chosen appropriately, local gauge invariance can be recovered.

In the section B, the *local* gauge-invariance was acquired by replacing ∂_{μ} with

$$\mathbf{D}_{\mu} = \partial_{\mu} + \mathcal{A}_{\mu}$$

(see (82)), where $A_{\mu} = (e/i\hbar c)A_{\mu}$, and $A_{\mu}(x)$ is the electromagnetic potential (and termed a *connection form* in mathematics). The operator D_{μ} is called the *covariant derivative*.

Thus, when the original Lagrangian is not locally gauge invariant, the *principle of local gauge invariance* requires a new gauge field to be introduced in order to acquire local gauge invariance, and the Lagrangian is to be altered by replacing the partial derivative with the *covariant* derivative including the gauge field. This is the *Weyl's gauge principle*.

In mathematical terms, suppose that we have a group \mathcal{G} of transformations and an element $g(x) \in \mathcal{G}$ (for $x \in M$ with M a space where ψ is defined) and that the wave function $\psi(x)$ is transformed as $\psi' = g(x)\psi$. In the previous example, $g(x) = e^{i\alpha(x)}$ and the group is $\mathcal{G} = U(1)$. Introducing the gauge field A allows us to define a covariant derivative $D = \partial + A$ as a generalization of the partial derivative ∂ that transforms as $gD = g(\partial + A) = (\partial' + A')g$. If we operates the right hand side on ψ , we obtain $(\partial' + A')g\psi = (\partial' + A')\psi' = D'\psi'$ where $D' = \partial' + A'$. Thus, we obtain $D'\psi' = g D\psi$, showing that $D\psi$ transforms in the same way as ψ itself.

In dynamical systems which evolve with the time t, such as the present case of fluid flows, replacement is to be made only for the t derivative: $\partial_t \rightarrow D_t = \partial_t + A(x)$.

B Gauge transformations

Local gauge transformations are considered, and then invariance of the operator D_t is presented, from §7.6 of [3].

B.1 Local gauge transformations

Suppose that there are two Eulerian coordinate frames F and F'. We consider a transformation of the position coordinate x of F to x' of another frame F'. Suppose that the transformation is given by

LGT:
$$x'(x,t) = x + \xi(x,t), \quad t' = t.$$
 (83)

This *LGT* is regarded as a local gauge transformation between two non-inertial frames of reference F and F'. In fact, it means that the position coordinate $X_a(a,t)$ of a fluid particle a in the frame F is transformed to a new position coordinate X'_a given by $X'_a(X_a,t) = X_a(t) + \xi(X_a,t)$ in the frame F'. Therefore, its velocity $v = (d/dt)X_a(t)$ is transformed to

$$(\mathbf{d}/\mathbf{d}t)\boldsymbol{\xi}_{a} = \partial_{t}\boldsymbol{\xi} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{\xi}\Big|_{\boldsymbol{x}=X_{a}},$$
 (85)

where $\boldsymbol{\xi}_a \equiv \boldsymbol{\xi}(X_a, t)$. This is a transformation between two coordinate values of the *same* particle described by two different frames of reference F and F'.⁷ Physically, two vectors X_a and X'_a denote the same point, represented by the same value of Lagrange coordinate \boldsymbol{a} . Namely, we are considering a gauge transformation between two reference frames.

According to the transformation (83), the time derivative ∂_t and space derivative $\partial_k = \partial/\partial x^k$ in the frame Fare related to the derivatives $\partial_{t'}$ and $\partial'_k = \partial/\partial x'_k$ of F' as follows:

$$\partial_t = \partial_{t'} + (\partial_t \boldsymbol{\xi}) \cdot \nabla', \qquad \nabla' = (\partial'_k), \quad (86)$$

$$\partial_k = \partial'_k + \partial_k \xi_l \,\partial'_l, \qquad \quad \partial'_k = \partial/\partial x'_k.$$
 (87)

The transformation *LGT* is also called a local *translational* transformation.

B.2 Gauge invariance of **D**_t

The operator $D_t \equiv \partial_t + (\boldsymbol{v} \cdot \nabla)$ is invariant with respect to *LGT*: *i.e.* $D_t = D'_t$. In fact from (84) and (87), we have

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⁷In this respect, present transformation is different from the variation considered in §7.5.3, although the expression $\mathbf{x}'(\mathbf{x}, t) = \mathbf{x} + \boldsymbol{\xi}$ is the same for the two cases.

where $v = v' - \mathrm{d}\xi/\mathrm{d}t$ is used. The last term is rewritten as

$$\left(-(\mathrm{d}\boldsymbol{\xi}/\mathrm{d}t)+\boldsymbol{v}\cdot\nabla\boldsymbol{\xi}\right)\cdot\nabla' = -\partial_t\boldsymbol{\xi}\cdot\nabla' = \partial_{t'}-\partial_t,$$

by using (85) and (86). Hence, we have

$$\mathbf{D}_t = \partial_t + \boldsymbol{v} \cdot \nabla = \partial_{t'} + \boldsymbol{v}' \cdot \nabla'. \tag{88}$$

This means that the operator D_t satisfies the invariance with respect to *LGT*, *i.e.* the translation symmetry. Thus, the operator D_t is the *covariant derivative* in the sense of gauge theory [3], in place of the partial derivative ∂_t .

The particle labels $a^i(t, \boldsymbol{x})$ (i = 1, 2, 3) are scalars, and move together with the material particle with the velocity $\boldsymbol{v} = \partial_t X(t, \boldsymbol{a})$. Hence, at any time, we have $D_t a^i = \partial_t a^i + (\boldsymbol{v} \cdot \nabla)a^i = 0$.Writing the particle position as $\boldsymbol{x} = X(t, \boldsymbol{a})$, we have

$$D_t \boldsymbol{x} = D_t X(t, \boldsymbol{a}(t, \boldsymbol{x})) = \partial_t X(t, \boldsymbol{a}) + D_t \boldsymbol{a} \cdot \nabla_{\boldsymbol{a}} X$$

$$= \partial_\tau X(\tau, \boldsymbol{a}) \equiv \boldsymbol{v}.$$
(89)

where $(\nabla_a X) = (\partial X^k / \partial a^l)$. Thus, we have the equality:

$$\partial_{\tau} = \mathbf{D}_t = \partial_t + (\boldsymbol{v} \cdot \nabla),$$

which was used already in §7.5.1. Velocity v can be defined by $D_t x$. In fact, operating $D'_{t'}$ on the equation (83) and using $D'_{t'} = D_t$, we obtain

$$\boldsymbol{v}' = \mathbf{D}'_{t'}\boldsymbol{x}' = \mathbf{D}_t(\boldsymbol{x} + \boldsymbol{\xi}) = \boldsymbol{v} + \mathbf{D}_t\boldsymbol{\xi}.$$
 (90)

This is consistent with (84). Thus, the particle velocity is defined by $v(x,t) = D_t x$.

B.3 Gauge transformation (general)

Suppose that we have a group \mathcal{G} , and consider the following transformation by its element $g \in \mathcal{G}$.

(a) A field u(x) is defined on a manifold M. Suppose that the coordinate $x \in M$ is transformed to x' = gx by $g \in \mathcal{G}$, and the field u to u' defined by u'g = gu simultaneously. Then we have

$$u'(x') \equiv u'g x = gu x = gu(x) \tag{91}$$

This means that the field u is transformed in the same way as the coordinate x.

(b) Next, suppose that a field of group element g(x)(where $g \in \mathcal{G}$) is defined at each point $x \in M$, and u(x) is transformed according to u'g = gu.

In addition, in place of the partial derivative ∂_t (with respect to time), we define a covariant derivative $D_t = \partial_t + A$ by introducing a gauge field A.

Its transformation is assumed to be given by

$$\mathsf{D}_{t'}' g = g \, \mathsf{D}_t,$$

where $D'_{t'} = \partial' + A'$ and $\partial' = \partial/\partial t'$. Operating the left side $D'_{t'}g$ on u, we obtain $D'_{t'}gu = D'_{t'}u'g$. Equating this to the right side $g D_t u$, we have

$$\mathbf{D}_{t'}' g u = \mathbf{D}_{t'}' u' g \equiv (\mathbf{D}_t u)' g = g \mathbf{D}_t u.$$
(92)

Hence, $D_t u$ is transformed in the same way as u.

In the example of the previous section where g is LGT, by setting u to be the particle coordinate x and gu the transformed coordinate x', $D_t u$ corresponds to the velocity v, and the equation (92) can be written as

$$\boldsymbol{v}'(\boldsymbol{x}') = \mathrm{D}'_{t'}\boldsymbol{x}' = g\,\mathrm{D}_t\boldsymbol{x} = g\,\boldsymbol{v}(\boldsymbol{x}),$$

where gv is defined by $v + D_t \xi$ by (90). We consider this kind of transformations below.

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